

# DIFFERENTIAL CALCULUS ON $q$ -DEFORMED LIGHT-CONE

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## Abstract

We propose the “short” version of  $q$ -deformed differential calculus on the light-cone using twistor representation. The commutation relations between coordinates and momenta are obtained. The quasi-classical limit introduced gives an exact shape of the off-shell shifting.

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# 1 Introduction

The deformed differential calculus [1, 2] plays an important role in physical applications [3, 4]. In this connection the light-cone approach attracts the special interest, because it is employed for the description of massless particles, null-strings and null-membranes [5]. The usual formalism of the bi-covariant differential calculi [6, 7] and right (left) invariant differential calculi [8] are not applicable on the light-cone due to vanishing of the corresponding quantum determinant. That is the reason why the Maurer-Cartan forms cannot be defined here at all, and therefore it is not possible to build the deformed differential calculus on the light-cone in the standard way.

From general mathematical viewpoint some objects under consideration belong to the class of quantum semigroups [9] which consist of the standard quantum groups and ideals (for abstract semigroup theory see [10] and for application to supersymmetry see [11, 12]).

Here we construct a version of  $q$ -deformed differential calculus on the light-cone which is mostly close to the standard one [13, 14]. We therefore hope that the obtained calculus can be directly used in physical applications.

## 2 “Short” differential calculus

Let us remind the deformed differential calculi on the quantum plane  $E_q$  (2) [1]

$$xy = qyx. \quad (1)$$

In the standard case we have two solutions for  $E_q$  (2)

a)

$$\begin{cases} \delta xx &= q^2 x \delta x, \\ \delta xy &= qy \delta x + (q^2 - 1) x \delta y, \\ \delta yx &= qx \delta y, \\ \delta yy &= q^{-2} y \delta y; \end{cases} \quad (2)$$

b)

$$\begin{cases} \delta xx &= q^2 x \delta x, \\ \delta xy &= q^{-1} y \delta x, \\ \delta yx &= q^{-1} x \delta y + (q^{-2} - 1) y \delta x, \\ \delta yy &= q^{-2} y \delta y. \end{cases} \quad (3)$$

It is well-known [15] that there exists the automorphism of the quantum plane (1)

$$\begin{pmatrix} x \rightarrow y \\ y \rightarrow x \\ q \rightarrow q^{-1} \end{pmatrix} \quad (4)$$

which is stipulated by the following property of the  $R$ -matrix

$$R_q \rightarrow R_{q^{-1}}^{-1}. \quad (5)$$

If we introduce the evolution parameter [3] and consider the dynamics on  $E_q(2)$  where (2) are commutation relations between momentum and coordinate, then we come to very complicated and inappropriate conditions for the phase space. The reason lays in the second term of the second formula in (2). Such undesirable terms (so called “long” solutions) result in difficulties while deriving and exploiting of self-consistent Poisson bracket.

In search of “short” solutions we found the following ones on  $E_q(2)$  additionally to (2) and (3)

$$\begin{cases} \delta xx &= x\delta x, \\ \delta xy &= qy\delta x, \\ \delta yx &= q^{-1}x\delta y, \\ \delta yy &= y\delta y. \end{cases} \quad (6)$$

In formulas (2), (3) and (6)  $\delta$  is the standard exterior differential satisfying the classical Leibniz rule, lemma Poincare and having the properties  $\delta x\delta y = -q\delta y\delta x$ ,  $(\delta x)^2 = 0$ ,  $(\delta y)^2 = 0$ .

In contrast to the “long” solutions (2) and (3) which are transformed one to another by the automorphism (4), the “short” solution (6) is its fixed point.

### 3 The $q$ -deformed light-cone

Let us consider  $q$ -deformed null-vector in 4-dimensional Minkowski space which is described by the following  $(2 \times 2)$   $q$ -matrix

$$X = \begin{pmatrix} x^{1\dot{1}} & x^{1\dot{2}} \\ x^{2\dot{1}} & x^{2\dot{2}} \end{pmatrix}. \quad (7)$$

We define the  $q$ -deformed light-cone as

$$\det_{q^2} X = x^{1\dot{1}} x^{2\dot{2}} - q^2 x^{1\dot{2}} x^{2\dot{1}} = 0. \quad (8)$$

Due to multiplicativity of the quantum determinant the Lorentz transformations (represented by  $SL_q(2, C)$  matrices) do not change the light-cone condition (8).

The  $q$ -deformed components have the following commutation relations

$$\begin{cases} x^{1\dot{1}} x^{1\dot{2}} &= q^2 x^{1\dot{2}} x^{1\dot{1}}, \\ x^{1\dot{1}} x^{2\dot{1}} &= q^2 x^{2\dot{1}} x^{1\dot{1}}, \\ x^{1\dot{2}} x^{2\dot{1}} &= x^{2\dot{1}} x^{1\dot{2}}, \\ x^{1\dot{2}} x^{2\dot{2}} &= q^2 x^{2\dot{2}} x^{1\dot{2}}, \\ x^{2\dot{1}} x^{2\dot{2}} &= q^2 x^{2\dot{2}} x^{2\dot{1}}, \\ x^{1\dot{1}} x^{2\dot{2}} - x^{2\dot{2}} x^{1\dot{1}} &= (q^2 - q^{-2}) x^{1\dot{2}} x^{2\dot{1}}. \end{cases} \quad (9)$$

As usual [16, 17] a null vector has a twistor representation. In  $q$ -deformed case we introduce the twistors having  $q$ -deformed components

$$\varphi_q = (\varphi_q^A) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = 1, 2 \quad (10)$$

where  $x$  and  $y$  satisfy (1).

The  $q$ -deformed Levi-Chivita tensor is

$$\epsilon_q = (\epsilon_q^{AB}) = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix} \quad (11)$$

and has the property

$$\epsilon_q^{AB} \epsilon_{q,BA} = q + q^{-1}. \quad (12)$$

The  $q$ -antisymmetry of  $\epsilon_q$  leads to

$$\varphi_q^A \varphi_q^B \epsilon_{q,AB} = \varphi_q^A \varphi_{q,A} = 0.$$

The complex conjugated  $q$ -deformed twistor is

$$\overline{\varphi}_q = (\overline{\varphi}_q^{\dot{A}}) = (\overline{x}, \overline{y}).$$

We define here the involution  $x \rightarrow \bar{x}$  as the standard Hermitian conjugation.

The  $q$ -null-vector in terms of  $q$ -twistor components has the standard form

$$X^{A\dot{A}} = \varphi_q^A \bar{\varphi}_q^{\dot{A}} = \begin{pmatrix} x\bar{x} & x\bar{y} \\ y\bar{x} & y\bar{y} \end{pmatrix}. \quad (13)$$

The light-cone condition (8) in terms of  $q$ -twistor components is

$$\det_{q^2} X = x\bar{x}y\bar{y} - q^2 x\bar{y}y\bar{x} = 0. \quad (14)$$

So we have two copies of the quantum plane  $(x, y)$  and  $(\bar{x}, \bar{y})$ . Now we need such commutation relations between components of the  $q$ -twistor and its conjugate which satisfy (9) and the standard involution.

From (1) and its conjugate we easily obtain the condition

$$\bar{q} = q^{-1} = \exp(-ih) \quad (15)$$

or  $|q| = 1$ .

Let other commutation relations have the general form

$$\begin{cases} x\bar{x} &= q^n \bar{x}x, \\ x\bar{y} &= q^m \bar{y}x, \\ y\bar{x} &= q^k \bar{x}y, \\ y\bar{y} &= q^l \bar{y}y, \end{cases} \quad (16)$$

where  $n, m, k, l$  are arbitrary constants which should be determined from (9) and involution. So we have the following independent equations

$$\begin{cases} m - n &= 1, \\ n - k &= 1, \\ l - k &= 1. \end{cases}$$

Then

$$\begin{cases} x\bar{x} &= q^n \bar{x}x, \\ x\bar{y} &= q^{n+1} \bar{y}x, \\ y\bar{x} &= q^{n-1} \bar{x}y, \\ y\bar{y} &= q^n \bar{y}y. \end{cases} \quad (17)$$

Now we consider the reality condition, as the consequence of the possibility to reduce the dimension of  $q$ -deformed null-vector up to three where it

is real. So we derive  $n = 0$ , and the commutation relations between twistor components become

$$\begin{cases} xy = qyx, \\ x\bar{y} = q\bar{y}x, \\ \bar{x}y = qy\bar{x}, \\ \overline{xy} = q\overline{yx}, \\ \bar{x}x = x\bar{x}, \\ \bar{y}y = y\bar{y}. \end{cases} \quad (18)$$

Then we can find the “short” version of  $q$ -deformed differential calculus on twistor components

$$\begin{cases} \delta xx = x\delta x, \\ \delta x\bar{x} = \bar{x}\delta x, \\ \delta xy = qy\delta x, \\ \delta x\bar{y} = q\bar{y}\delta x, \end{cases} \quad \begin{cases} \delta\bar{x}x = x\delta\bar{x}, \\ \delta\bar{x}\bar{x} = \bar{x}\delta\bar{x}, \\ \delta\bar{x}y = qy\delta\bar{x}, \\ \delta\bar{x}\bar{y} = q\bar{y}\delta\bar{x}, \end{cases} \quad (19)$$

$$\begin{cases} \delta yx = q^{-1}x\delta y, \\ \delta y\bar{x} = q^{-1}\bar{x}\delta y, \\ \delta yy = y\delta y, \\ \delta y\bar{y} = \bar{y}\delta y, \end{cases} \quad \begin{cases} \delta\bar{y}x = q^{-1}x\delta\bar{y}, \\ \delta\bar{y}\bar{x} = q^{-1}\bar{x}\delta\bar{y}, \\ \delta\bar{y}y = y\delta\bar{y}, \\ \delta\bar{y}\bar{y} = \bar{y}\delta\bar{y}, \end{cases} \quad (20)$$

where  $(\delta x)^2 = (\delta\bar{x})^2 = (\delta y)^2 = (\delta\bar{y})^2 = 0$ .

The commutation relations between the differentials themselves take the form

$$\begin{cases} \delta x\delta\bar{x} = -\delta\bar{x}\delta x, \\ \delta x\delta y = -q\delta y\delta x, \\ \delta x\delta\bar{y} = -q\delta\bar{y}\delta x, \\ \delta y\delta x = -q^{-1}\delta x\delta y, \\ \delta y\delta\bar{x} = -q^{-1}\delta\bar{x}\delta y, \\ \delta y\delta\bar{y} = -\delta\bar{y}\delta y. \end{cases} \quad (21)$$

Using the obtained  $q$ -deformed differential calculus on  $q$ -twistors we can build  $q$ -deformed differential calculus on any composite objects, as  $q$ -deformed null-vectors and tensors.

So for  $q$ -deformed null-vector we obtain

$$\begin{cases} \delta x^{1\dot{1}} x^{1\dot{1}} = x^{1\dot{1}} \delta x^{1\dot{1}}, \\ \delta x^{1\dot{1}} x^{1\dot{2}} = q^2 x^{1\dot{2}} \delta x^{1\dot{1}}, \\ \delta x^{1\dot{1}} x^{2\dot{1}} = q^2 x^{2\dot{1}} \delta x^{1\dot{1}}, \\ \delta x^{1\dot{1}} x^{2\dot{2}} = q^4 x^{2\dot{2}} \delta x^{1\dot{1}}, \end{cases} \begin{cases} \delta x^{1\dot{2}} x^{1\dot{2}} = x^{1\dot{2}} \delta x^{1\dot{2}}, \\ \delta x^{1\dot{2}} x^{2\dot{1}} = x^{2\dot{1}} \delta x^{1\dot{2}}, \\ \delta x^{1\dot{2}} x^{1\dot{1}} = q^{-2} x^{1\dot{1}} \delta x^{1\dot{2}}, \\ \delta x^{1\dot{2}} x^{2\dot{2}} = q^2 x^{2\dot{2}} \delta x^{1\dot{2}}, \end{cases} \quad (22)$$

$$\begin{cases} \delta x^{2\dot{1}} x^{2\dot{1}} = x^{2\dot{1}} \delta x^{2\dot{1}}, \\ \delta x^{2\dot{1}} x^{1\dot{1}} = q^{-2} x^{1\dot{1}} \delta x^{2\dot{1}}, \\ \delta x^{2\dot{1}} x^{1\dot{2}} = x^{1\dot{2}} \delta x^{2\dot{1}}, \\ \delta x^{2\dot{1}} x^{2\dot{2}} = q^2 x^{2\dot{2}} \delta x^{2\dot{1}}, \end{cases} \begin{cases} \delta x^{2\dot{2}} x^{2\dot{2}} = x^{2\dot{2}} \delta x^{2\dot{2}}, \\ \delta x^{2\dot{2}} x^{1\dot{1}} = q^{-4} x^{1\dot{1}} \delta x^{2\dot{2}}, \\ \delta x^{2\dot{2}} x^{1\dot{2}} = q^{-2} x^{1\dot{2}} \delta x^{2\dot{2}}, \\ \delta x^{2\dot{2}} x^{2\dot{1}} = q^{-2} x^{2\dot{1}} \delta x^{2\dot{2}}, \end{cases} \quad (23)$$

and

$$\begin{aligned} \delta x^{1\dot{1}} \delta x^{1\dot{2}} &= -q^2 \delta x^{1\dot{2}} \delta x^{1\dot{1}}, \\ \delta x^{1\dot{1}} \delta x^{2\dot{1}} &= -q^2 \delta x^{2\dot{1}} \delta x^{1\dot{1}}, \\ \delta x^{1\dot{1}} \delta x^{2\dot{2}} &= -q^4 \delta x^{2\dot{2}} \delta x^{1\dot{1}}, \\ \delta x^{1\dot{2}} \delta x^{2\dot{1}} &= -\delta x^{2\dot{1}} \delta x^{1\dot{2}}, \\ \delta x^{1\dot{2}} \delta x^{2\dot{2}} &= -q^2 \delta x^{2\dot{2}} \delta x^{1\dot{2}}, \\ \delta x^{2\dot{1}} \delta x^{2\dot{2}} &= -q^2 \delta x^{2\dot{2}} \delta x^{2\dot{1}}. \end{aligned} \quad (24)$$

From the above formulas we can find the commutation relations for co-ordinates and derivatives

$$\begin{cases} \frac{\partial}{\partial x^{1\dot{1}}} x^{1\dot{1}} = 1 + x^{1\dot{1}} \frac{\partial}{\partial x^{1\dot{1}}}, \\ \frac{\partial}{\partial x^{1\dot{1}}} x^{1\dot{2}} = q^{-2} x^{1\dot{2}} \frac{\partial}{\partial x^{1\dot{1}}}, \\ \frac{\partial}{\partial x^{1\dot{1}}} x^{2\dot{1}} = q^{-2} x^{2\dot{1}} \frac{\partial}{\partial x^{1\dot{1}}}, \\ \frac{\partial}{\partial x^{1\dot{1}}} x^{2\dot{2}} = q^{-4} x^{2\dot{2}} \frac{\partial}{\partial x^{1\dot{1}}}, \end{cases} \begin{cases} \frac{\partial}{\partial x^{1\dot{2}}} x^{1\dot{1}} = q^2 x^{1\dot{1}} \frac{\partial}{\partial x^{1\dot{2}}}, \\ \frac{\partial}{\partial x^{1\dot{2}}} x^{1\dot{2}} = 1 + x^{1\dot{2}} \frac{\partial}{\partial x^{1\dot{2}}}, \\ \frac{\partial}{\partial x^{1\dot{2}}} x^{2\dot{1}} = x^{2\dot{1}} \frac{\partial}{\partial x^{1\dot{2}}}, \\ \frac{\partial}{\partial x^{1\dot{2}}} x^{2\dot{2}} = q^{-2} x^{2\dot{2}} \frac{\partial}{\partial x^{1\dot{2}}}, \end{cases} \quad (25)$$

$$\begin{cases} \frac{\partial}{\partial x^{2\dot{1}}} x^{1\dot{1}} = q^2 x^{1\dot{1}} \frac{\partial}{\partial x^{2\dot{1}}}, \\ \frac{\partial}{\partial x^{2\dot{1}}} x^{2\dot{1}} = 1 + x^{2\dot{1}} \frac{\partial}{\partial x^{2\dot{1}}}, \\ \frac{\partial}{\partial x^{2\dot{1}}} x^{1\dot{2}} = x^{1\dot{2}} \frac{\partial}{\partial x^{2\dot{1}}}, \\ \frac{\partial}{\partial x^{2\dot{1}}} x^{2\dot{2}} = q^{-2} x^{2\dot{2}} \frac{\partial}{\partial x^{2\dot{1}}}, \end{cases} \begin{cases} \frac{\partial}{\partial x^{2\dot{2}}} x^{1\dot{1}} = q^4 x^{1\dot{1}} \frac{\partial}{\partial x^{2\dot{2}}}, \\ \frac{\partial}{\partial x^{2\dot{2}}} x^{1\dot{2}} = q^2 x^{1\dot{2}} \frac{\partial}{\partial x^{2\dot{2}}}, \\ \frac{\partial}{\partial x^{2\dot{2}}} x^{2\dot{1}} = q^2 x^{2\dot{1}} \frac{\partial}{\partial x^{2\dot{2}}}, \\ \frac{\partial}{\partial x^{2\dot{2}}} x^{2\dot{2}} = 1 + x^{2\dot{2}} \frac{\partial}{\partial x^{2\dot{2}}}, \end{cases} \quad (26)$$

and between the derivatives

$$\begin{aligned}
\frac{\partial}{\partial x^{1\dot{1}}} \frac{\partial}{\partial x^{1\dot{2}}} &= q^2 \frac{\partial}{\partial x^{1\dot{2}}} \frac{\partial}{\partial x^{1\dot{1}}}, \\
\frac{\partial}{\partial x^{1\dot{1}}} \frac{\partial}{\partial x^{2\dot{1}}} &= q^2 \frac{\partial}{\partial x^{2\dot{1}}} \frac{\partial}{\partial x^{1\dot{1}}}, \\
\frac{\partial}{\partial x^{1\dot{1}}} \frac{\partial}{\partial x^{2\dot{2}}} &= q^4 \frac{\partial}{\partial x^{2\dot{2}}} \frac{\partial}{\partial x^{1\dot{1}}}, \\
\frac{\partial}{\partial x^{1\dot{2}}} \frac{\partial}{\partial x^{2\dot{1}}} &= \frac{\partial}{\partial x^{2\dot{1}}} \frac{\partial}{\partial x^{1\dot{2}}}, \\
\frac{\partial}{\partial x^{1\dot{2}}} \frac{\partial}{\partial x^{2\dot{2}}} &= q^2 \frac{\partial}{\partial x^{2\dot{2}}} \frac{\partial}{\partial x^{1\dot{2}}}, \\
\frac{\partial}{\partial x^{2\dot{1}}} \frac{\partial}{\partial x^{2\dot{2}}} &= q^2 \frac{\partial}{\partial x^{2\dot{2}}} \frac{\partial}{\partial x^{2\dot{1}}}.
\end{aligned} \tag{27}$$

By analogy we can write the commutation relation between  $q$ -deformed differentials and derivatives.

The above relations are consistent with the following condition

$$\delta (\det_{q^2} X) = 0. \tag{28}$$

## 4 Momenta and $q$ -D'Alembertian

Let us introduce  $q$ -matrix for momenta

$$P_q = \begin{pmatrix} P_{1\dot{1}} & P_{1\dot{2}} \\ P_{2\dot{1}} & P_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} -i \frac{\partial}{\partial x^{1\dot{1}}} & -i \frac{\partial}{\partial x^{1\dot{2}}} \\ -i \frac{\partial}{\partial x^{2\dot{1}}} & -i \frac{\partial}{\partial x^{2\dot{2}}} \end{pmatrix} \tag{29}$$

with commutation relations determined by (27).

The  $q$ -D'Alembertian is defined by

$$\square_{q^2} = -\det_{q^2} P_q = \frac{\partial}{\partial x^{1\dot{1}}} \frac{\partial}{\partial x^{2\dot{2}}} - q^2 \frac{\partial}{\partial x^{1\dot{2}}} \frac{\partial}{\partial x^{2\dot{1}}}. \tag{30}$$

The light-cone condition (8) leads to the similar condition for momenta

$$\square_{q^2} = 0. \tag{31}$$



In the classical limit  $q = \exp(ih) \rightarrow 1$  we decompose the  $q$ -D'Alembertian as follows

$$\begin{aligned}\square_{q^2} &= \square - (q^2 - 1) \frac{\partial}{\partial x^{1\dot{2}}} \frac{\partial}{\partial x^{2\dot{1}}} \\ &= \square - 2ihP_{1\dot{2}}P_{2\dot{1}} = 0,\end{aligned}\tag{32}$$

where  $\square$  is the ordinary D'Alembertian.

The second term in (32) gives us a new way of the off-shell approximation and is responsible for its exact shape. In contrast to the standard picture [18] the momenta entering into the additional off-shell term in (32) commute.

## 5 Conclusion

We have proposed a version of the differential calculus on  $q$ -deformed light-cone which can be applied to description of the dynamics of the massless quantum particles,  $q$ -deformed null-strings and null-membranes.

Using the obtained  $q$ -deformed differential calculus on  $q$ -twistors we have the possibility to construct the corresponding calculi on  $q$ -tensors of any rank.

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